Final Exam

Physics 425/525

Multiple Choice (circle or otherwise indicate the correct answer)

1. (5pts) A (singly charged) deuteron of mass 3.343×10^{-27} kg is accelerated in a cyclotron with a magnetic field 0.1 T. What is the cyclotron frequency?
   a. 4.79×10^{5} /sec = 0.479 MHz
   b. 4.79×10^{6} /sec = 4.79 MHz
   c. 4.79×10^{7} /sec = 47.9 MHz
   d. 4.79×10^{8} /sec = 479 MHz

$$\omega_c = \frac{\left(1.602\times10^{-19}\,\mathrm{C}\right)\left(0.1\ \mathrm{Vsec/m^2}\right)}{3.343\times10^{-27}\,\mathrm{kg}} = 0.479\times10^{7}\ \mathrm{sec}^{-1}$$

2. (5pts) Recall $P_1(x) = x$ and $P_2(x) = (3x^2-1)/2$. What are $\int_{-1}^{1} x^2 dx$ and $\int_{-1}^{1}\left[(3x^2-1)^2/4\right]dx$?
   a. $2/3;\ 2/5$
   b. $2/3;\ 2/7$
   c. $2/5;\ 2/7$
   d. $2/7;\ 2/9$

$$\int_{-1}^{1} P_1^2(x)\,dx = \frac{2}{2\cdot1+1} \qquad \int_{-1}^{1} P_2^2(x)\,dx = \frac{2}{2\cdot2+1}$$

Of course, you could also just perform the integrals!

3. (5pts) What is $\int_{-1}^{1}\left[(3x^3-x)/2\right]dx$?
   a. 2/7
   b. 2/5
   c. 0
   d. 1

$$\int_{-1}^{1} P_1(x)P_2(x)\,dx = 0$$

Again, you could just do the integral and get the same result.

4. (5pts) Two parallel wires of length 1 m separated by 10 cm are excited by a current 1 A in the same direction. Assuming the magnetic field is constant on each wire, the total force(s) on the wires are
   a. 2×10^{-6} Nt repulsive along the line between the wires
   b. 2×10^{-6} Nt attractive along the line between the wires

c. 2×10^{-3} Nt repulsive along the line between the wires

d. 2×10^{-3} Nt attractive along the line between the wires

The magnetic force on each wire is

$$F_{mag} = IBl = I\frac{\mu_0 I}{2\pi s}l = \frac{4\pi\times10^{-7}\left(\text{Nt}/\text{A}^2\right)}{2\pi}\left(1\ \text{A}\right)^2\frac{1\ \text{m}}{0.1\ \text{m}} = 2\times10^{-6}\ \text{Nt}$$

A right hand rule analysis shows the force between the wires is attractive.

5. (5pts) Suppose $\boldsymbol{B} = \left[B_0\sigma_s\left(1-\exp\left(-s^2/2\sigma_s^2\right)\right)/s\right]\hat{\phi}$ where s is the cylindrical radial coordinate and $\hat{\phi}$ is the cylindrical unit vector. What is $\boldsymbol{J}(s)$?
   a. $\left(B_0/\mu_0\sigma_s\right)^2\exp\left(-s^2/2\sigma_s^2\right)\hat{z}$
   b. $\left(B_0/\mu_0\sigma_s\right)\exp\left(-s^2/2\sigma_s^2\right)\hat{z}$
   c. $\left(B_0/\mu_0\sigma_s\right)\left(1-\exp\left(-s^2/2\sigma_s^2\right)\right)\hat{z}$
   d. $\left(sB_0/\mu_0\sigma_s^2\right)\exp\left(-s^2/2\sigma_s^2\right)\hat{z}$

$$\boldsymbol{J} = \nabla\times\left(\boldsymbol{B}/\mu_0\right) = \frac{\hat{z}}{s}\frac{\partial}{\partial s}\left[\frac{B_0\sigma_s}{\mu_0}\left(1-\exp\left(-s^2/2\sigma_s^2\right)\right)\right] = \frac{\hat{z}}{s}\frac{B_0\sigma_s}{\mu_0}\exp\left(-s^2/2\sigma_s^2\right)\frac{2s}{2\sigma_s^2}$$

6. (10 pts) For each vector field in the right hand list, associate the proper SI unit for the field found in the left hand list by filling in the blank with the correct letter.

a. V sec/m^2 (T)	***E***.................d................
b. A/m	***D***.................c................
c. C/m^2	***H***.................b................
d. Nt/C	***B***.................a................
	M.................b................

7. (10 pts) For each boundary condition in the right hand list, associate the proper static Maxwell Equation that generates the boundary condition in the left hand list by filling in the blank with the correct letter.

a. $\nabla\cdot\boldsymbol{D} = \rho_f$	$E_{above}^{\parallel} = E_{below}^{\parallel}$ ....c Al......................
b. $\nabla\times\boldsymbol{H} = \boldsymbol{J}_f$	$B_{above}^{\perp} = B_{below}^{\perp}$ ....d Gp......................
c. $\nabla\times\boldsymbol{E} = 0$	$\varepsilon_{above}E_{above}^{\perp} - \varepsilon_{below}E_{below}^{\perp} = \sigma_f$ ....a Gp......................
d. $\nabla\cdot\boldsymbol{B} = 0$	$\left(\boldsymbol{B}_{above}^{\parallel}/\mu_{above}\right)-\left(\boldsymbol{B}_{below}^{\parallel}/\mu_{below}\right) = \boldsymbol{K}_f\times\hat{n}$ ....b Al......................

Extra Credit (4 pts): Identify whether each boundary condition is generated by a Gaussian pillbox (Gp) or Ampere loop (Al) argument by adding Gp or Al to each answer.

8. (10 pts) Show the vector field $\boldsymbol{v}_1 = x\hat{x} + y\hat{y} + z\hat{z}$ is curl-free and find a scalar potential for it. Show the vector field $\boldsymbol{v}_2 = x\hat{x} - y\hat{y}$ is divergence-free and find a vector potential for it.

$$\nabla \times \boldsymbol{v}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0-0)\hat{x} + (0-0)\hat{y} + (0-0)\hat{z} = 0$$

$$\phi(\boldsymbol{r}) = \int_0^{\boldsymbol{r}} \boldsymbol{v}_1 \cdot d\boldsymbol{l} = \int_0^{\boldsymbol{r}} (xdx + ydy + zdz) = \int_0^{\boldsymbol{r}} d\left(\frac{x^2+y^2+z^2}{2}\right) = \frac{r^2}{2}$$

$$\nabla \cdot \boldsymbol{v}_2 = \frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} = 1 - 1 = 0$$

$$V_x = \int_0^1 txtdt = \frac{x}{3} \qquad V_y = -\int_0^1 tytdt = -\frac{y}{3}$$

$$\boldsymbol{A} = \boldsymbol{V} \times \boldsymbol{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{x}{3} & -\dfrac{y}{3} & 0 \\ x & y & z \end{vmatrix} = -\frac{yz}{3}\hat{x} + \frac{xz}{3}\hat{y} + \frac{2xy}{3}\hat{z}$$

9. (20 pts) A conducting sphere of radius R is charged with a charge Q. It is surrounded by linear dielectric material of permittivity ε stretching to infinity.
   a. What is the electric field in each of the regions $r < R$ and $r > R$?

   For $r > R$

$$4\pi r^2 D_r = Q$$

$$\boldsymbol{D} = \frac{Q}{4\pi r^2}\hat{r}$$

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon r^2}\hat{r}$$

   For $r < R$, $\boldsymbol{E}$ is zero.

   b. What is the total energy of the configuration, including that energy needed to polarize the material?

$$U_e = \frac{1}{2}\int \boldsymbol{D}\cdot\boldsymbol{E}dV = \frac{1}{2}\frac{Q}{4\pi}\frac{Q}{4\pi\varepsilon}\int_R^\infty \frac{1}{r^4}r^2 dr\,(4\pi)$$

$$= \frac{Q^2}{8\pi\varepsilon R}$$

   c. Extra Credit (5 pts.) Show the result in (b) is consistent with the potential formula

$$U_e = \frac{1}{2}\int \rho(\boldsymbol{r})V(\boldsymbol{r})dV$$

The potential for the electric field is

$$V(\boldsymbol{r}) = \frac{Q}{4\pi\varepsilon r}.$$

On the conductor, its value is

$$\frac{Q}{4\pi\varepsilon R}.$$

So, because all the charge is at the conductor potential

$$U_e = \frac{1}{2}\int \rho(\boldsymbol{r})V(\boldsymbol{r})dV = \frac{1}{2}Q\frac{Q}{4\pi\varepsilon R}.$$

10. (25 pts) Consider an ideal infinite solenoid with turns/length n and radius R. Assume the magnetic field is uniform inside the solenoid and vanishes outside the solenoid.
    a. With vacuum conditions inside the solenoid, what is the magnetic field ***B*** when the coil current is *I*?
    By Ampere's Law

$$Bl = \mu_0 nIl$$
$$B = \mu_0 nI$$
$$\boldsymbol{B} = \mu_0 nI\hat{z}$$

    b. What is the magnetic force on the coil per unit length of coil (magnitude and direction)? (Assume the coil current feels the full magnetic field at the coil location and not the average of ***B***/2. Also ignore the fact that the coil current direction is slightly off the $\hat{\phi}$ direction.)
    The magnitude of the force/length is simply BI . As the direction for the coil current is $\hat{\phi}$, the direction for the force is $\hat{r} = \hat{\phi}\times\hat{z}$. So

$$\frac{\boldsymbol{F}}{\text{length}} = \mu_0 nI^2\hat{r}.$$

    There is no net force on the coil, but it does tend to want to expand itself!

    c. Now assume the solenoid contains linear material with magnetic susceptibility χ_m . What are the magnetic field ***B*** and magnetization ***M***, again in terms of the exciting current *I*?
    Via Ampere's Law with materials

$$Hl = nIl$$
$$H = nI$$
$$\boldsymbol{H} = nI\hat{z}$$

So

$$\boldsymbol{B} = \mu\boldsymbol{H} = \mu_0\left(1+\chi_m\right)\boldsymbol{H} = \mu_0\left(1+\chi_m\right)nI\hat{z}$$
$$\boldsymbol{M} = \chi_m nI\hat{z}$$